

2.4) General Cylindrical Metric

Start with the metric:

$$ds^2 = c^2 d\tau^2 = e^{-2\alpha/z} c^2 dt^2 - dr^2 - r^2 d\phi^2 - e^{2\alpha/z} dz^2 \quad (1)$$

The Lagrangian is:

$$L = T = \frac{1}{2} mc^2 = \frac{m}{2} \left(e^{-2\alpha/z} c^2 \left(\frac{dt}{d\tau} \right)^2 - \left(\frac{dr}{d\tau} \right)^2 - r^2 \left(\frac{d\phi}{d\tau} \right)^2 - e^{2\alpha/z} \left(\frac{dz}{d\tau} \right)^2 \right) \quad (2)$$

So:

$$m \left(\left(\frac{dr}{d\tau} \right)^2 + e^{2\alpha/z} \left(\frac{dz}{d\tau} \right)^2 \right) = mc^2 e^{-2\alpha/z} \left(\frac{dt}{d\tau} \right)^2 - mc^2 - mr^2 \left(\frac{d\phi}{d\tau} \right)^2 \quad (3)$$

Multiply through by $e^{-2\alpha/z}$:

$$\begin{aligned} m \left(e^{-2\alpha/z} \left(\frac{dr}{d\tau} \right)^2 + \left(\frac{dz}{d\tau} \right)^2 \right) &= mc^2 e^{-2\alpha/z} \left(\frac{dt}{d\tau} \right)^2 \\ &\quad - e^{-2\alpha/z} \left(mc^2 + mr^2 \left(\frac{d\phi}{d\tau} \right)^2 \right) \\ &= \frac{E^2}{mc^2} - e^{-2\alpha/z} \left(mc^2 + \frac{L^2}{mr^2} \right) \end{aligned} \quad (4)$$

Therefore:

$$m \frac{dd^2}{d\tau^2} = \frac{E^2}{mc^2} - e^{-2\alpha/z} \left(mc^2 + \frac{L^2}{mr^2} \right) \quad (5)$$

Use

$$\boxed{dd^2 = dz^2 + e^{-2\alpha/z} dr^2} \quad (6)$$

Now use:

$$\frac{dd}{d\tau} = \frac{dd}{d\phi} \frac{d\phi}{d\tau} = \left(\frac{L}{mr^2} \right) \frac{dd}{d\phi} \quad - (7)$$

So

$$\frac{dd}{d\phi} = r^2 \left(\frac{1}{b^2} - e^{-2.0/z} \left(\frac{1}{a^2} + \frac{1}{z^2} \right) \right)^{1/2} \quad - (8)$$

$$\frac{d\phi}{dd} = \frac{1}{r^2} \left(\frac{1}{b^2} - e^{-2.0/z} \left(\frac{1}{a^2} + \frac{1}{z^2} \right) \right)^{-1/2} \quad - (9)$$

$$\phi = \int \frac{1}{r^2} \left(\frac{1}{b^2} - e^{-2.0/z} \left(\frac{1}{a^2} + \frac{1}{z^2} \right) \right)^{-1/2} dd \quad - (10)$$

$$dd = \left(dz^2 + e^{-2.0/z} dz^2 \right)^{1/2} \quad - (11)$$

Light deflection is:

$$\Delta\phi = 2 \int_{R_0}^{\infty} \frac{1}{r^2} \left(\frac{1}{b^2} - e^{-2.0/z} \left(\frac{1}{a^2} + \frac{1}{z^2} \right) \right)^{-1/2} dd \quad - (12)$$