

278(S) : Some Theories of the Dielectric Loss

From previous notes:

$$n^2 \omega_1^2 = \omega^2 + \omega_2^2 - 2\omega_1 \omega_2 \cos 2\theta \quad - (1)$$

and

$$\omega = \omega_1 + \omega_2 \quad - (2)$$

where n is the refractive index, ω the incident frequency, ω_1 the refracted frequency and ω_2 the reflected frequency. Here $\theta/2$ is the incident and reflected angle.

Now assume that:

$$n^2 = n'^2 = \frac{1}{2} \left(\epsilon' + (\epsilon'^2 + \epsilon''^2)^{1/2} \right) \quad - (3)$$

where ϵ' is the dielectric dispersion and ϵ'' the dielectric loss. The power absorption coefficient is:

$$d(\omega) = \frac{\omega \epsilon''}{n' c} \quad - (4)$$

Eqs (1) and (2) can be solved for ω_1 as a function of ω and ω_2 as a function of ω . The result depends on a theory of the dielectric loss and dispersion.

Debye Theory (1913)

Here:

$$\epsilon(\omega) = \epsilon_\infty + \frac{(\epsilon_0 - \epsilon_\infty)}{1 - i\omega\tau} \quad - (5)$$

where τ is the Debye relaxation time.

2) So:

$$\epsilon' + i\epsilon'' = \epsilon_\infty + (\epsilon_0 - \epsilon_\infty) \frac{(1 + i\omega\tau)}{1 + \omega^2\tau^2} \quad - (6)$$

i.e. $\epsilon' = \epsilon_\infty + \frac{(\epsilon_0 - \epsilon_\infty)}{1 + \omega^2\tau^2} \quad - (7)$

$$\epsilon'' = \frac{(\epsilon_0 - \epsilon_\infty)\omega\tau}{1 + \omega^2\tau^2} \quad - (8)$$

For eqs. (4) and (8), $d(\omega)$ goes to a constant as the frequency goes to infinity. Therefore the Debye theory fails qualitatively at high frequencies.

Memory Function Theory (Omnia Opera 20)

The spectrum is given by the continued fraction:

$$\tilde{c}(p) = \frac{c(0)}{p + \frac{k_0(0)}{p + \frac{k_1(p)}{p + \dots}}} \quad - (9)$$

where

$$p = -i\omega \quad - (10)$$

The permittivity and dielectric loss are built up from various truncations of eq. (9). The first order truncation is:

$$K_1(s) = K_1(0) e^{-\gamma t} \quad - (11)$$

it is clear the memory function is assumed to decay exponentially. This means that:

$$K_1 = \frac{K_1(0)}{p + \gamma} \quad - (12)$$

So:

$$\tilde{C}(p) = \frac{C(0)}{p + \frac{K_0(0)}{p + \frac{K_1(0)}{p + \gamma}}} \quad - (13)$$

i.e.

$$\tilde{C}(p) = \frac{C(0)(p^2 + p\gamma + K_1(0))}{p^3 + p(K_0(0) + K_1(0)) + p^2\gamma + \gamma K_0(0)} \quad - (14)$$

where

$$p = -i\omega \quad - (15)$$

So:

$$\tilde{C}(p) = \frac{C(0)}{D} (A - i\omega\gamma)(B - i\omega C) \quad - (16)$$

where

$$\begin{aligned} A &= K_1(0) - \omega^2 \\ B &= \gamma(K_0(0) - \omega^2) \\ C &= \omega^2 - (K_0(0) + K_1(0)) \end{aligned} \quad - (17)$$

$$D = \gamma^2(K_0(0) - \omega^2)^2 + \omega^2(\omega^2 - (K_0(0) + K_1(0)))^2$$

For Laplace transform theory & power

4) asymptotic coefficient is:

$$d(\omega) = \omega^2 \operatorname{Re} \tilde{c}(p) - (18)$$

The Debye decay is recovered by:

$$\tilde{c}(p) = \frac{c(0)}{p + \gamma} - (19)$$

$$= c(0) \frac{(i\omega + \gamma)}{\gamma^2 + \omega^2}$$

So:

$$\omega \operatorname{Re} \tilde{c}(p) = \frac{\omega \gamma c(0)}{\gamma^2 + \omega^2} - (20)$$

and

$$\frac{1}{\omega} \operatorname{Im} \tilde{c}(p) = \frac{\gamma c(0)}{\gamma^2 + \omega^2} - (21)$$

Therefore

$$\epsilon''(\omega) = (t_0 - t_\infty) \omega \operatorname{Re} \tilde{c}(p) - (22)$$

and

$$\epsilon'(\omega) = \epsilon_\infty + \frac{(t_0 - t_\infty)}{\omega} \operatorname{Im} \tilde{c}(p) - (23)$$

For the two variable memory function theory:

$$\epsilon''(\omega) = \frac{(t_0 - t_\infty)}{\omega} \omega \gamma k_0(0) k_1(0) - (24)$$

and

$$\epsilon'(\omega) = \epsilon_\infty - \frac{(t_0 - t_\infty)}{\omega} c(0) (\omega A C + \gamma B) - (25)$$

Refraction

The relevant frequency is ω_1 , and ω_2 is eliminated between eqs (1) and (2). Then n^2 is calculated from eqs (3), (24) and (25), using the frequency ω_1 , so:

$$n^2 = f(\omega_1) - (26)$$

$$\omega_2 = \omega - \omega_1 - (27)$$

> Therefore:

$$f(\omega_1) \omega_1^2 = \omega^2 + (\omega - \omega_1)^2 - 2\omega_1(\omega - \omega_1) \cos 2\theta$$

and so ω can be expressed as a function of ω_1 and θ by solving the quadratic equation (28).

Square of the Refractive Index

If it is assumed in eq. (3) that:

$$n^2 = (n' + in'')^2 = n'^2 - n''^2 + 2in'n'' \quad (29)$$

then

$$\boxed{\begin{aligned} \text{Real } n^2 &= n'^2 - n''^2 \\ &= \epsilon_r' \end{aligned}} \quad (30)$$

So eq. (1) simplifies to:

$$\epsilon_r' \omega_1^2 = \omega^2 + \omega_2^2 - 2\omega_1\omega_2 \cos 2\theta \quad (31)$$

where:

$$\omega = \omega_1 + \omega_2 \quad (32)$$

b) So eq. (28) becomes:

$$\epsilon_r' \omega_1^2 = \omega^2 + (\omega - \omega_1)^2 - 2\omega_1(\omega - \omega_1) \cos 2\theta \quad - (33)$$

If we use the Debye theory:

$$\epsilon_r' = \epsilon_\infty + \frac{(\epsilon_0 - \epsilon_\infty)}{1 + \omega_1^2 \tau^2} \quad - (34)$$

and the two variable memory function theory:

$$\epsilon_r' = \epsilon_\infty - \omega_1 (\epsilon_0 - \epsilon_\infty) \frac{(\gamma B + AC)}{D} \quad - (35)$$

Here:

$$A = \kappa_1(0) - \omega_1^2$$

$$B = \gamma (\kappa_0(0) - \omega_1^2) \quad - (36)$$

$$C = \omega_1^2 - (\kappa_0(0) + \kappa_1(0))$$

$$D = \gamma^2 (\kappa_0(0) - \omega_1^2)^2 + \omega_1^2 (\omega_1^2 - (\kappa_0(0) + \kappa_1(0)))^2$$

so ω can be expressed in terms of ω_1 .

This procedure can be repeated for higher continued fraction theories, synthesizing any type of spectrum:

$$\bar{c}(p) = \frac{c(0)}{p + \frac{\kappa_0(0)}{p + \frac{\kappa_1(0)}{p + \frac{\kappa_2(0)}{p + \dots}}}} \quad - (34)$$

where : $\epsilon_r \propto \frac{\text{Im } \bar{c}(p)}{\omega} - (35)$

For each feature in the spectrum there will be features in both the refracted frequency ω_1 and the reflected frequency ω_2 .
